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Canonical Quantization of Field Theories with Boundaries

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Abstract The problem of canonical quantization of singular systems in a finite volume is studied by analysing a non-relativistic field theory. Firstly, we take the boundary conditions (BCs) as primary Dirac constraints. The quantization is performed canonically using Dirac's procedure. Then, we quantize this model canonically in the classical solution space. We show that these two different quantization schemes are equivalent although they start from different settings.

Keywords Constraints · Dirac method · Boundary conditions · Classical solutions

1 Introduction

It is well known that to study field theories in a finite volume, one should consider not only the equations of motion but also the boundary conditions (BCs). BCs are usually the combinations of the field variables and their various derivatives [1–5] which are valid only on the boundaries and are expected to be held all the times. In the Hamiltonian formalism, those BCs are the combinations of the canonical variables in phase space, i.e, the fields and their conjugate momenta (or their spatial derivatives). In Dirac's language, these BCs are the constraints in the phase space. However, such constraints have the different origins compared to the traditional Dirac' context. Due to BCs, one cannot quantize the system

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consistently in the whole space because on the boundaries those BCs are inconsistent with the usual canonical commutation relations [1–5].

This problem has been analysed by taking the BCs as primary Dirac constraints in [1]. To illustrate the ideas, the authors study two examples. Later, the models contain both the intrinsic constraints and BCs are studied [6, 7]. The methods the authors employed are also Dirac' theory. In [8, 9], we propose a new method to study the analogous problems during the investigation of opens strings in the D-brane background at the presence of B fields. We find that it is possible to find the commutation relations among the Fourier modes firstly and then get the Poisson brackets among the field variables.

In this paper, we shall analyse a model which contains both intrinsic constraints and BCs. To illustrate our ideas, we shall study a non-relativistic field, Schrödinger field (which is described by a singular Lagrangian) in a finite volume. So, from the point of view of origination, this model has two kinds of constraints, one kind is due to the singularities of Lagrangian, the other kind is BCs. What is more interesting is that these two kinds of constraints entangled on the boundaries.

The organization of this paper is as follows: in Sect. 2, we shall take the BCs as primary constraints and quantize the model canonically in a finite volume using Dirac's procedure. Then in the Sect. 3, we analyse this model in the classical solutions space. We turn the Lagrangian from a continuous integration to a discrete summation by means of redefining new dynamical variables in the classical solution space. The commutation relations among the new variables are obtained and then the Poisson brackets among the original variables are determined. Some further discussions and remarks will be given in the last section.

2 Dirac's Procedure

We start from Schrödinger field living in an infinite volume. The basic field variables are $\psi(x,t)$ and their Hermitian conjugate $\psi^\dagger(x,t)$. The action is (for the sake of simplicity, we set $m=\hbar=1$, and we restrict ourself to $1+1$ dimension)

$$\begin{aligned} S=\int_{t_1}^{t_2} dt L = \frac{1}{2}\int_{t_1}^{t_2} dt \int_{-\infty}^{+\infty} dx\, i\psi^\dagger(x,t)\partial_t\psi(x,t) \\ - i\partial_t\psi^\dagger(x,t)\psi(x,t) - \partial_x\psi^\dagger(x,t)\partial_x\psi(x,t) \end{aligned} \tag{1}$$

where ∂_t and ∂_x denote $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ respectively. L is the Lagrangian

$$L=\frac{1}{2}\int_{-\infty}^{+\infty} dx\, i\psi^\dagger(x,t)\partial_t\psi(x,t) - i\partial_t\psi^\dagger(x,t)\psi(x,t) - \partial_x\psi^\dagger(x,t)\partial_x\psi(x,t). \tag{2}$$

Variation of the action with respects to $\psi(x,t)$ and $\psi^\dagger(x,t)$ will lead to the equations of motion. They are the free Schrödinger equation and its Hermitian conjugation

$$i\partial_t\psi(x,t)+\frac{1}{2}\partial_x\partial_x\psi(x,t)=0, \qquad -i\partial_t\psi^\dagger(x,t)+\frac{1}{2}\partial_x\partial_x\psi^\dagger(x,t)=0. \tag{3}$$

The Lagrangian (2) is a constrained one in the Dirac's terminology [10]. It is easy to find that there are only two primary constraints

$$\phi_1^{(0)}(x,t)=\Pi(x,t)-\frac{i}{2}\psi^\dagger(x,t)\approx 0, \qquad \phi_2^{(0)}(x,t)=\Pi^\dagger(x,t)+\frac{i}{2}\psi(x,t)\approx 0 \tag{4}$$

where $\Pi(x,t)$ and $\Pi^\dagger(x,t)$ are canonical momenta corresponding to variables $\psi(x,t)$ and $\psi^\dagger(x,t)$ respectively.

The Dirac brackets can be calculated in the standard way. The results can be found in [11]. Here we only list the results:

$$\begin{aligned}\{\psi(x,t),\ \psi^\dagger(x',t)\}_{IDB} &= -i\delta(x-x'),\\ \{\psi(x,t),\ \Pi(x',t)\}_{IDB} &= \frac{1}{2}\delta(x-x').\end{aligned} \tag{5}$$

(The reason why we label as $\{\,,\,\}_{IDB}$ instead of $\{\,,\,\}_{DB}$ will be obvious shortly after.)

The quantization procedure is complete once we take the following substitution:

$$\begin{aligned}&\{\,,\,\}_{DBI} \to \tfrac{1}{i}[\,,\,],\\ &\psi \to \hat{\psi}, \qquad \psi^\dagger \to \hat{\psi}^\dagger, \qquad \Pi \to \hat{\Pi}, \qquad \Pi^\dagger \to \hat{\Pi}^\dagger.\end{aligned} \tag{6}$$

However, when the field is confined in a finite volume, the above results (5) must be modified since they conflict with BCs on the boundary generally.

The action for the field in a finite volume is

$$\begin{aligned}S = \int_{t_1}^{t_2} dt L = \frac{1}{2}\int_{t_1}^{t_2} dt \int_0^\pi dx i\psi^\dagger(x,t)\partial_t\psi(x,t)\\ - i\partial_t\psi^\dagger(x,t)\psi(x,t) - \partial_x\psi^\dagger(x,t)\partial_x\psi(x,t).\end{aligned} \tag{7}$$

Compared with the action (1), the spiral integral is confined in a finite volume, i.e, $x \in [0,\pi]$.

Variation of the action with respect to $\psi(x,t)$ and $\psi^\dagger(x,t)$ leads to both the equations of motion (3) and the BCs. They are Neumann BCs

$$\partial_x\psi(x,t)|_{x=0,\pi} = 0, \qquad \partial_x\psi^\dagger(x,t)|_{x=0,\pi} = 0 \tag{8}$$

and Dirichlet ones

$$\delta\psi(x,t)|_{x=0,\pi} = 0, \qquad \delta\psi^\dagger(x,t)|_{x=0,\pi} = 0. \tag{9}$$

For the case of Dirichlet boundary conditions, they generally can be rewritten as

$$\psi(x,t)|_{x=0,\pi} = c, \qquad \psi^\dagger(x,t)|_{x=0,\pi} = c^\dagger \tag{10}$$

where c is a constant. Without losing any generalities, we set $c=0$.

Because of the BCs (8) and (10), one cannot impose (5) in the whole space because they conflict with BCs on the boundaries. As a result, commutators (5) need modify so as to make them be consistent with the BCs on the boundaries. In [1], the authors treat the BCs as primary constraint and show that BCs form the second class constraints. Then, the Dirac's procedure is applied. Our model (7) is more complicated since this model contains both intrinsic constraints (4) and BCs (8), (10). We shall follow the same idea, take the BCs (8), (10) as Dirac primary constraints in this section. As an example, we only consider the Dirichlet BCs (10).

Now besides the constraints (4), there are two additional constraints (10). We note that although the BCs are only valid on the boundary, they can be extended into the neighborhood of the boundaries using Dirac delta function. It means that we can rewrite them in the form as

$$\phi_3^{(0)} = \int_0^\pi dx\psi(x,t)\delta(x-B) \approx 0, \qquad \phi_4^{(0)} = \int_0^\pi dx\psi^\dagger(x,t)\delta(x-B) \approx 0 \tag{11}$$

where B stands for boundaries, $B = 0, \pi$.

It can be checked that there are no secondary constraints. The matrix of mutual Poisson brackets of primary constraints $\phi_i^{(0)}$, $i = 1, 2, 3, 4$, C_{ij} is defined as

$$C_{ij} = \{\phi_i^{(0)}(x,t),\ \phi_j^{(0)}(x',t)\}. \tag{12}$$

The explicit expression for C_{ij} is

$$C = \begin{pmatrix} 0 & -i\delta(x-x') & -\delta(x-B) & 0 \\ i\delta(x-x') & 0 & 0 & -\delta(x-B) \\ \delta(x-B) & 0 & 0 & 0 \\ 0 & \delta(x-B) & 0 & 0 \end{pmatrix}. \tag{13}$$

From the explicit expression of matrix C, we know that the intrinsic constraints ($\phi_1^{(0)} \approx 0$ and $\phi_2^{(0)} \approx 0$) and the BCs ($\phi_3^{(0)} \approx 0$ and $\phi_4^{(0)} \approx 0$) are entangled and formed second class constraints.

It is difficult to compute the inverse of C directly. In order to simplify this problem, we prefer to do this in two steps: first, we construct intermediate Dirac brackets which correspond to the constraints $\phi_1^{(0)} \approx 0$, $\phi_2^{(0)} \approx 0$, then bracket which correspond to the remaining constraints $\phi_3^{(0)} \approx 0$, $\phi_4^{(0)} \approx 0$. Consistency of doing so is guaranteed by a known theorem [12]. In fact, the intermediate Dirac brackets is nothing but (5), this is the reason why we label Dirac brackets previously as $\{\,,\,\}_{IDB}$ instead of $\{\,,\,\}_{DB}$. The intermediate Dirac brackets of remaining constraints $\phi_3^{(0)} \approx 0$, $\phi_4^{(0)} \approx 0$ are

$$\{\phi_3^{(0)}, \phi_4^{(0)}\}_{IDB} = -i\int dx dx' \delta(x-B)\delta(x'-B)\delta(x-x'). \tag{14}$$

Using the equality

$$\delta(x-x') = \lim_{\epsilon\to 0} \frac{1}{\epsilon\sqrt{\pi}} e^{-(x-x')^2/\epsilon^2} \tag{15}$$

the result of (14) can be obtained

$$\{\phi_3^{(0)},\ \phi_4^{(0)}\}_{IDB} = -\frac{i}{\epsilon\sqrt{\pi}}. \tag{16}$$

The matrix of intermediate Dirac brackets corresponding to $\phi_3^{(0)}$, $\phi_4^{(0)}$ is

$$\Delta = \begin{pmatrix} 0 & -\frac{i}{\epsilon\sqrt{\pi}} \\ \frac{i}{\epsilon\sqrt{\pi}} & 0 \end{pmatrix}. \tag{17}$$

This matrix can be easily inverted, and the final expression for the Dirac brackets is

$$\{A(x,t), B(x',t)\}_{DB} = \{A(x,t), B(x',t)\}_{IDB} - \{A(x,t), \theta_i\}_{IDB}\Delta_{ij}^{-1}\{\theta_j, B(x',t)\}_{IDB} \tag{18}$$

in which θ_i are $\phi_3^{(0)}$ and $\phi_4^{(0)}$ now. The canonical commutators can be gotten from the above equation. We list our final results

$$\begin{aligned}\{\psi(x,t),\ \psi^\dagger(x',t)\}_{DB} &= -i\delta(x-x') + i\epsilon\sqrt{\pi}\delta(x-B)\delta(x'-B),\\ \{\psi(x,t),\ \Pi(x',t)\}_{DB} &= \frac{1}{2}\delta(x-x') - \frac{\epsilon\sqrt{\pi}}{2}\delta(x-B)\delta(x'-B).\end{aligned} \tag{19}$$

Others are vanishing. The appearance of the regularization parameter ϵ in the above equation seems uncomfortable, but it is necessary to keep the two terms on the right side to be of the same order [1].

Our final results (20) is right the $\delta(x-x')$ if we only consider the bulk, i.e, $x,\ x' \in (0,\pi)$, however, on the boundaries, they are consistent with Dirichlet BCs, using equality (15) one can easily verify that

$$\{\psi(x,t),\ \psi^\dagger(x',t)\}_{DB}|_{x=B} = 0. \tag{20}$$

For this reason, we label the right hand side of (20) as $\delta_B(x-x')$, i.e,

$$\begin{aligned}\{\psi(x,t), \psi^\dagger(x',t)\}_{DB} &= -i\delta_B(x-x'),\\ \{\psi(x,t), \Pi(x',t)\}_{DB} &= \frac{1}{2}\delta_B(x-x').\end{aligned} \tag{21}$$

The canonical quantization procedure is straightforward, the only modification to (6) is that the subscript is DB instead of IDB.

3 Quantization in the Classical Solution Space

In the previous section, we take the BCs as primary Dirac constraints and apply Dirac's procedure to modify Dirac brackets so as to make them consistent with BCs on the boundaries. In [8, 9], we propose a new method to quantize in the classical solution space during the study of open strings in the D-brane background with NS B field. From the point of view of field theories, this problem is the one how to treat BCs in field theories. But that model contains no intrinsic constraints. Here we show that this method still work in the case of both the intrinsic and BCs are presented.

The solutions which satisfy both the equations of motion (3) and Dirichlet BCs (10) are

$$\begin{aligned}\psi(x,t) &= \sum_{n=0}^{\infty}(a_n - a_{-n})\exp\left(-i\frac{n^2}{2}t\right)\sin nx,\\ \psi^\dagger(x,t) &= \sum_{n=0}^{\infty}(a_n^\dagger - a_{-n}^\dagger)\exp\left(i\frac{n^2}{2}t\right)\sin nx\end{aligned} \tag{22}$$

where a_n and $a_n^\dagger$ are Fourier modes.

Introduce dynamical variables

$$\begin{aligned}a_n(t) &= a_n e^{-i\frac{n^2}{2}t}, &\quad a_n^\dagger(t) &= a_n^\dagger e^{i\frac{n^2}{2}t},\\ A_n(t) &= a_n(t) - a_{-n}(t), &\quad A_n^\dagger(t) &= a_n^\dagger(t) - a_{-n}^\dagger(t).\end{aligned} \tag{23}$$

We rewrite (22) as

$$\psi(x,t)=\sum_{n=0}^{\infty}A_n(t)\sin nx,\qquad \psi^{\dagger}(x,t)=\sum_{n=0}^{\infty}A_n^{\dagger}(t)\sin nx. \tag{24}$$

Substitute (24) into the action (7) and integrate with respect to spatial variable, we get

$$S=\int dt\frac{i\pi}{2}\sum_{n=0}^{\infty}[A_n^{\dagger}(t)\dot{A}_n(t)-A_n(t)\dot{A}_n^{\dagger}(t)]-\frac{\pi}{2}\sum_{n=0}^{\infty}n^2A_n^{\dagger}(t)A_n(t). \tag{25}$$

After the substitution, we turn the action (1) from a continuous integration to a discrete summation and the dynamical variables are changed from $\psi(x,t)$, $\psi^{\dagger}(x,t)$ to $A_n(t)$ and $A_n^{\dagger}(t)$. Now, we introduce conjugate momenta Π_n and $\Pi_n^{\dagger}$ to $A_n(t)$ and $A_n^{\dagger}(t)$

$$\begin{aligned}\Pi_n(t)&=\frac{\delta S}{\delta\dot{A}_n(t)}=\frac{i\pi}{2}A_n^{\dagger}(t),\\ \Pi_n^{\dagger}(t)&=\frac{\delta S}{\delta\dot{A}_n^{\dagger}(t)}=-\frac{i\pi}{2}A_n(t)\end{aligned} \tag{26}$$

and impose the standard classical Poisson brackets among variables $A_n(t)$, $A_n^{\dagger}(t)$, Π_n and $\Pi_n^{\dagger}$,

$$\{A_m(t),\Pi_n(t)\}=\delta_{mn},\qquad \{A_m^{\dagger}(t),\Pi_n^{\dagger}(t)\}=\delta_{mn}. \tag{27}$$

Now, the classical Poisson brackets among variables $\psi(x,t)$ and $\psi^{\dagger}(x,t)$ can be computed. Taking into the (27) into consideration, we get

$$\begin{aligned}\{\psi(x,t),\psi^{\dagger}(x',t)\}&=\sum_{m,n=0}^{\infty}\{A_n(t),A_m^{\dagger}(t)\}\sin nx\sin mx'\\ &=\frac{-i}{\pi}\sum_{n=-\infty}^{\infty}\sin nx\sin nx'.\end{aligned} \tag{28}$$

The right hand sides of the above equations are proportional to the Fourier expansion of delta function $\delta(x-x')$ defined on the interval $[0,\ \pi]$

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\sin nx\sin nx'=\delta(x-x'). \tag{29}$$

What is more important is that it has vanishing value on the boundaries, i.e,

$$\sum_{n=-\infty}^{\infty}\sin nx\sin nx'|_{x,x'=B}=0.$$

For this reason, we also label it as

$$\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\sin nx\sin nx'=\delta_B(x-x'). \tag{30}$$

So the right hand sides of (28) can be written as

$$\begin{aligned}
\{\psi(x,t),\psi^\dagger(x',t)\} &= -i\delta_B(x-x'),\\
\{\psi(x,t),\Pi(x',t)\} &= \frac{1}{2}\delta_B(x-x').
\end{aligned} \tag{31}$$

The quantization in the reduced phase space is straightforward. It can be achieved simply by the substitution

$$\begin{aligned}
&\{\,,\,\} \to \frac{1}{i}[\,,\,],\\
&\psi \to \hat{\psi}, \qquad \psi^\dagger \to \hat{\psi}^\dagger, \qquad \Pi \to \hat{\Pi}, \qquad \Pi^\dagger \to \hat{\Pi}^\dagger.
\end{aligned} \tag{32}$$

The results (20) obtained in the previous section are equivalent to (28). Since they are all delta function $\delta(x-x')$ in the interior and all have vanishing values on the boundaries.

4 Conclusions and Remarks

To have a better understanding of fields in a finite volume at the quantum level, it is necessary to quantize them. Due to boundary conditions, one cannot impose the standard Poisson brackets among the canonical variables directly. For the models of both the intrinsic constraints and boundary conditions are contained, the situation will be more complicated.

In this paper, we study this problem by analysing free $1+1$ dimensional Schrödinger field. Compared with the previous work [1], our model is more general since it contains both intrinsic constraints and BCs. Following [1], we take the BCs as Dirac primary constraints. It is shown that BCs entangle with the intrinsic constraints on the boundaries and form the second class constraints. In order to quantize this model canonically, the calculations of Dirac brackets are inevitable. Based on a theorem [12], the calculation are greatly simplified. We construct the intermediate Dirac brackets $\{\,,\,\}_{IDB}$ firstly, the final Dirac brackets $\{\,,\,\}_{DB}$ are calculated based on the intermediate ones.

Then, we study this model in the classical solution space. The dynamical variables are transformed from original variables to Fourier modes and the original Lagrangian is transformed from a continuous integration to discrete summation in this classical solution space. We first get the commutation relations among the Fourier modes and then the commutation relations among the original variables can be obtained. The generalization of our methods to systems contain gauge fields are in consideration.

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References

1. Sheikh-Jabbari, M.M., Shirzad, A.: Eur. Phys. J. C **19**, 383 (2001)
2. Ardalan, F., Arfaei, H., Sheikh-Jabbari, M.M.: Nucl. Phys. B **576**, 578 (2000)
3. Chu, C.S., Ho, P.M.: Nucl. Phys. B **550**, 151 (1999)
4. Chu, C.S., Ho, P.M.: Nucl. Phys. B **568**, 447 (2000)
5. Dehghani, M., Shirzad, A.: Eur. Phys. J. C **48**, 315–325 (2006)
6. Jing, J.: Eur. Phys. J. C **39**, 123 (2005)

7. Jing, J.: Phys. Rev. D **71**, 025023 (2005)
8. Jing, J., Long, Z.-W.: Phys. Rev. D **72**, 126002 (2005)
9. Jing, J.: Phys. Rev. D **73**, 086001 (2006)
10. Dirac, P.A.M.: Lecture Notes on Quantum Mechanics. Yeshiva University, New York (1964)
11. Li, Z.-P., Jiang, J.-H.: Symmetries in Constrained Canonical Systems. Science, Beijing (2002)
12. Gitman, D.M., Tyutin, I.V.: Quantization of Fields with Constraints. Springer, Berlin (1990)